# Groups generated by a finite Engel set

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**Abstract**

A subset $S$ of a group $G$ is called an Engel set if, for all $x, y \in S$, there is a non-negative integer $n = n(x, y)$ such that $[x,\, _n y] = 1$. In this paper we are interested in finding conditions for a group generated by a finite Engel set to be nilpotent. In particular, we focus our investigation on groups generated by an Engel set of size two.

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## 1 Introduction

A subset $S$ of a group $G$ is called an Engel set if, for all $x, y \in S$, there is a non-negative integer $n = n(x, y)$ such that $[x,\, _n y] = 1$. It is known that, for a group $G$ satisfying Max-$ab$, a normal subset $S \subseteq G$ is an Engel set if and only if it is contained in the Fitting subgroup of $G$ (see [7], Theorem 7.23; see

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also [1]) and so, in this case, $\langle S\rangle$ is nilpotent whenever $S$ is finite. However, a group generated by a finite Engel set is not necessarily nilpotent: Golod's examples show that there exist infinite non-nilpotent groups generated by an Engel set with three or more elements (see [5]). Furthermore, if $S$ is an Engel set of size three, then an easier example of a non-nilpotent group generated by $S$ is the wreath product of the alternating group of degree 5 with the cyclic group of order 3: it has a *presentation of type* $(r, s, t)$ (see [3]), i.e. $S = \{a, b, c\}$ where $\langle a, b\rangle$ is nilpotent of class $r$, $\langle a, c\rangle$ is nilpotent of class $s$ and $\langle b, c\rangle$ is nilpotent of class $t$. All these groups are not soluble, but the nilpotency does not hold even in the soluble case. In [3] it was shown that every group with a presentation of type (1,2,2) is soluble of length at most 3 and that there are non-nilpotent groups of this type.

In this paper, we first get that any nilpotent-by-abelian group generated by a finite Engel set is nilpotent and then we focus on groups generated by an Engel set of size two. In particular, we prove that such a group is nilpotent whenever it is abelian-by-(nilpotent of class 2). This is the best possible result in the soluble case. In fact, we construct by GAP (see [4]) a non-nilpotent counterexample which is abelian-by-(nilpotent of class 3). On the other hand, some of the counterexamples in [3], mentioned above, are abelian-by-(nilpotent of class 2) and generated by an Engel set of size three.

## 2 Groups that are Nilpotent-by-Abelian

We start with a result that is certainly already known. It generalizes, for metabelian groups, two basic properties of commutators.

**Lemma 2.1.** *Let $G$ be a metabelian group and $x, y, z$ be elements of $G$. For all positive integers $n$, we have:*

*(i)* $[x^{-1}, {}_n y] = [x, {}_n y]^{-x^{-1}}$;

*(ii)* $[xy, {}_n z] = [x, {}_n z][x, {}_n z, y][y, {}_n z]$.

*Proof.* Since $G$ is metabelian, every $g$ in $G$ induces on $G'$ an endomorphism $-1+g$ that maps $u$ to $u^{-1}u^g$, and any two of these commute. We thus have:

$$[x^{-1}, {}_n y] = ([x, y]^{-x^{-1}})^{(-1+y)^{n-1}} = [x, y]^{-(-1+y)^{n-1}x^{-1}} = [x, {}_n y]^{-x^{-1}}.$$

The proof of *(ii)* is similar. $\square$

As a consequence of Lemma 2.1, we get:

**Lemma 2.2.** *If $G$ is a metabelian group generated by an Engel set $S$, then any $x \in S$ is a left Engel element. In particular, $G$ is locally nilpotent.*

*Proof.* Take a finite subset of $S$, say $T = \{x_1, \dots, x_r\}$, and suppose $[x_i, {}_n x_j] = 1$ for all $1 \leq i, j \leq r$. By the previous lemma, every $x_i$ is a left $n$-Engel element in $G$. Then $(-1 + x_i)^n = 0$. It follows that any product in the endomorphisms $-1 + x_i$ of weight $(n-1)r + 1$ is trivial. Hence $\langle T \rangle$ is nilpotent of class at most $(n-1)r + 2$. This proves that $G$ is locally nilpotent. □

For a finite Engel set, we then obtain the following:

**Theorem 2.3.** *Let $G$ be a nilpotent-by-abelian group generated by a finite Engel set. Then $G$ is nilpotent.*

*Proof.* If $N$ is a normal nilpotent subgroup of $G$ such that $G/N$ is abelian, then $G/N'$ is nilpotent by Lemma 2.2 and so $G$ is nilpotent by a well-known result of P. Hall. □

# 3 Engel sets of size two

Let $G = \langle x, y \rangle$ be a group and assume that $\{x, y\}$ is an Engel set. Then $[x, {}_n y] = 1$ and $[y, {}_m x] = 1$ for some positive integers $n, m$. We also say that the elements $x$ and $y$ are *mutually Engel* and, whenever $n \geq m$, that they are *mutually $n$-Engel*. If $n = m = 2$, then $G$ is obviously nilpotent of class at most 2 and the nilpotency still holds for $n = 2$ and $m = 3$.

**Proposition 3.1.** *Let $G = \langle x, y \rangle$ be an arbitrary group such that $[x, y, y] = 1$ and $[y, x, x, x] = 1$. Then $G$ is nilpotent of class at most* 3.

*Proof.* By the Hall-Witt identity we have

$$[[y, x], x^{-1}, y]^x [x, y^{-1}, [y, x]]^y [y, [y, x]^{-1}, x]^{[y,x]} = 1,$$

from which it follows

$$[y, x, x^{-1}, y] = 1$$

since $[x, y^{-1}] = [x, y]^{-1}$ and $[y, [y, x]^{-1}] = [x, y, y]^{-1} = 1$. Then $[y, x, x, y] = 1$ and hence $[y, x, x] \in Z(G)$. Now $[x, y, y] = [y, x, x] = 1$ modulo $Z(G)$, so $G/Z(G)$ is nilpotent of class $\leq 2$ and $G$ is nilpotent of class $\leq 3$. □

However, as we will see in the next section, this is not true in general, even in the soluble case. We are therefore led to consider extra conditions for a group generated by an Engel set of size two to be nilpotent. In the sequel, we will turn our attention to groups which are abelian-by-(nilpotent of class 2).

Let $G$ be any abelian-by-(nilpotent of class 2) group generated by two mutually Engel elements $x$ and $y$. By assumption $[x, {}_n y] = 1$ and $[y, {}_n x] = 1$ for some $n$. Suppose, by way of contradiction, that $G$ is not nilpotent. Then

$G$ has a non-nilpotent finite image by Theorem 10.51 of [7] and so we may assume that $G$ is finite.

Using induction on the order of the group, we may assume that $G$ is a minimal counterexample. It follows that $G$ contains a unique minimal normal subgroup $A$ such that $G/A$ is nilpotent. As $G$ is not nilpotent there is a maximal subgroup $H$ that is not normal. On the other hand $G/A$ is nilpotent, therefore $A \not\leq H$ (otherwise $H/A \lhd G/A$ implies that $H \lhd G$). Thus $G = AH$. The group $A \cap H$ is normal in $G$ and $A \cap H < A$. The minimality of $A$ then forces $A \cap H = 1$.

Clearly, $A$ is an elementary abelian $p$-group for some prime $p$ and $H$ is nilpotent. Let $P$ be the Sylow $p$-subgroup of $H$. Then $AP/A \lhd G/A$ and so $AP$ is the Sylow $p$-subgroup of $G$. Since $AP$ is nilpotent, we have that $[A, AP] < A$ and by the minimality of $A$, the normal subgroup $[A, AP]$ must be trivial. Thus $[A, P] = 1$ and $P^G = P^{AH} = P^H = P$, that is $P \lhd G$. But $A \not\leq P$, hence $P = 1$ and $H$ is a Hall $p'$-subgroup of $G$.

**Lemma 3.2.** *Every nontrivial element of $Z(H)$ acts fixed point freely on $A$ by conjugation.*

*Proof.* For all $z \in Z(H)$ and $h \in H$, $C_A(z)^h = C_A(z)$ and thus $C_A(z) \lhd G$. As $\langle z \rangle$ cannot be normal in $G$, we get $C_A(z) = 1$ by minimality of $A$. □

The next lemma shows that $H$ is nilpotent of class 2 and that we can restrict our attention to $n = 3$.

**Lemma 3.3.** *Let $G = AH = \langle x, y \rangle$ be a minimal counterexample that is abelian-by-(nilpotent of class 2). Then $A = \gamma_3(G), [x, y, y, y] = 1$ and $[y, x, x, x] = 1$.*

*Proof.* Of course, $A \subseteq \gamma_3(G)$ by minimality of $A$. Let $q \neq p$ be a prime. Then any $q$-subgroup of $\gamma_3(G)$ is necessarily trivial. But $G/A$ is a $p'$-group, therefore $A = \gamma_3(G)$ and $H$ is nilpotent of class 2.

Assuming now $[x, {}_{n-1}y] \neq 1$, we will prove that $n \leq 3$. Let $y = ah$ where $a \in A, h \in H$, and suppose $n > 3$. We have $[x, y, y] \in A$ and $n - 2 \geq 2$, so that $[x, {}_{n-2}y]$ and $[x, {}_{n-2}y, y]$ lie in $A$. It follows that

$$[x, {}_{n-2}y, y^p] = [x, {}_{n-2}y, y]^p = 1.$$

Notice that $y^p = a_1 h^p$ with $a_1 \in A$ and $h = h^{\alpha p}$ for some integer $\alpha$. Thus

$$1 = [x, {}_{n-2}y, y^p] = [x, {}_{n-2}y, a_1 h^p] = [x, {}_{n-2}y, h^p]$$

and

$$1 = [x, {}_{n-2}y, h^{\alpha p}] = [x, {}_{n-2}y, h].$$

But then

$$1 = [x, {}_{n-2}y, ah] = [x, {}_{n-2}y, y],$$

that is a contradiction. □

We need one more preliminary lemma before proving our main result.

**Lemma 3.4.** *Let $x = ah, y = bk$ where $a, b \in A$ and $h, k \in H$. If $[x, y] = [h, k]$, then*

$$[a, k^{-1}] = [b, h^{-1}], \quad [a, h] = 1 \quad \textit{and} \quad [b, k] = 1,$$

*with $a \neq 1$ and $b \neq 1$.*

*Proof.* We have

$$[h, k] = [x, y] = [ah, bk] = [a, k]^h[h, k][h, b]^k.$$

This implies $[a, k]^h[h, b]^{h^{-1}kh} = 1$ and then $[a, k]^{k^{-1}} = [b, h]^{h^{-1}}$, or equivalently $[a, k^{-1}] = [b, h^{-1}]$.

As $G \neq H$ we must have that one of $a, b$ is nontrivial. Without loss of generality, we may assume $a \neq 1$. Clearly, $[y, x, x] \in A$ and $1 \neq [y, x] \in Z(H)$. Then $1 = [y, x, x, x] = [y, x, x, h]$ and

$$[x, h]^{[y,x]} = [x^{[y,x]}, h] = [[y, x, x]^{-1}x, h] = [x, h].$$

Thus $1 = [x, h, [y, x]] = [[a, h]^h, [y, x]] = [a, h, [y, x]]^h$, so $[a, h]$ is fixed by $[y, x]$. By Lemma 3.2 it follows that $[a, h] = 1$. As a consequence $b \neq 1$, otherwise $[a, k] = 1$ and $[a, [h, k]] = 1$. Arguing as for $a$, we then conclude that $[b, k] = 1$. □

**Theorem 3.5.** *Let $G$ be any abelian-by-(nilpotent of class 2) group generated by two mutually Engel elements $x$ and $y$. Then $G$ is nilpotent.*

*Proof.* Put $x = ah, y = bk$ where $a, b \in A$ and $h, k \in H$. Then $[x, y] = [h, k]c$ with $[h, k] \in Z(H)$ and for some $c \in A$. By Lemma 3.3 we know that

$$[x, y, y], [y, x, x] \in A \quad \text{and} \quad [x, y, y, y] = [y, x, x, x] = 1.$$

This gives

$$[x, y, y^p] = 1 \qquad \text{and} \qquad [x, y, x^p] = 1.$$

If $\langle x^p, y^p\rangle \cap A \neq 1$, the commutator $[x, y]$ commutes with a nontrivial element of $A$. Thus $[h, k] = 1$ by Lemma 3.2, and $[x, y] \in A$. Indeed $G' \leq A$ and $G$ is nilpotent by Lemma 2.2. Therefore $A \cap \langle x^p, y^p\rangle = 1$ and we may assume $H = \langle x^p, y^p\rangle$, since $\langle h, k\rangle \simeq \langle h, k\rangle A/A = \langle x^p, y^p\rangle A/A \simeq \langle x^p, y^p\rangle$. It follows that $c$ must be trivial. Then $1 \neq [x, y] = [h, k]$ and, by Lemma 3.4, we have

$$[a, k^{-1}] = [b, h^{-1}] \qquad \text{and} \qquad [a, h] = 1,$$

with $a \neq 1$.

Now, the Hall-Witt identity

$$[a, k^{-1}, h]^k[k, h^{-1}, a]^h[h, a^{-1}, k]^a = 1$$

implies

$$[a, k^{-1}, h]^k = [k, h^{-1}, a]^{-h}.$$

But $[k, h^{-1}, a]$ commutes with $h$, so $[[a, k^{-1}], h] = [[b, h^{-1}], h]$ commutes with $h^{k^{-1}}$. Then $[b, h, h]^{h^{-1}} = [b, h^{-1}, h]^{-1}$ commutes with $h^{k^{-1}}$, in particular $[b, h, h]$ commutes with $h^{k^{-1}h} = h^{k^{-1}}$. Hence $[b, h, h] \in C_A(h^{k^{-1}})$.

Let $B = C_A(h^{k^{-1}})$ and $K = \langle h, h^{k^{-1}} \rangle A$. Then $B \lhd K$ because $[h^{-1}, k] \in Z(H)$. If $q$ is the order of $h$, we also have $B = [b, h^q]B = [b, h]^q B$. However, the order of $[b, h]$ is coprime with $q$, thus $[b, h] \in B$ and $[a, k^{-1}] = [b, h^{-1}] \in B$. So $[a, k^{-1}, h^{k^{-1}}] = 1$ and $[k, a, h] = 1$. Finally, from

$$[a, k, h]^{k^{-1}} [k^{-1}, h^{-1}, a]^h [h, a^{-1}, k^{-1}]^a = 1,$$

it follows $[k, h, a] = 1$ which contradicts Lemma 3.2. □

When $x$ and $y$ are mutually 3-Engel elements, we get thanks to GAP that the group $G$ in Theorem 3.5 is nilpotent of class at most 8. In fact, using the ANU Nilpotent Quotient package of W. Nickel (see [6]), we can construct the largest nilpotent quotient of $G$ which is isomorphic to $G$.

Also notice that the theorem above can be extended to a group generated by more than two mutually Engel elements, provided that none of the generators has order divisible by 2 or 3.

**Corollary 3.6.** *Let $S$ be a finite Engel set and assume that $G = \langle S \rangle$ is abelian-by-(nilpotent of class 2). If every element in $S$ has order that is not divisible by 2 or 3, then $G$ is nilpotent.*

*Proof.* For all $x, y \in S$, the subgroup $\langle x, y \rangle$ is nilpotent by Theorem 3.5. Thus the claim follows by Proposition 1 of [3]. □

Using Theorem 3.5, we now present a criterion for nilpotency of a finite soluble group depending on information on its Sylow subgroups.

**Corollary 3.7.** *Let $G = \langle x, y \rangle$ be a finite soluble group with $x$ and $y$ mutually Engel elements. If all Sylow subgroups of $G$ are nilpotent of class $\leq 2$, then $G$ is nilpotent.*

*Proof.* Let $G$ be a counterexample of least possible order and let $N$ be a minimal normal subgroup of $G$. Then $G/N$ is nilpotent by minimality. Moreover, all Sylow subgroups of $G/N$ are nilpotent of class $\leq 2$, so that $G/N$ is nilpotent of class $\leq 2$. On the other hand $N$ is abelian, because $G$ is soluble. Hence $G$ is abelian-by-(nilpotent of class 2) and thus nilpotent by Theorem 3.5: a contradiction. □

## 4 Examples

Our first example shows that, for any positive integer $n$, there exists a group generated by two mutually $n$-Engel elements which are not $(n-1)$-Engel. This is the dihedral group of order $2^{n+1}$.

**Example 4.1.** Let consider $G = \langle x, y \mid x^2 = y^2 = 1, (xy)^{2^n} = 1\rangle$. If $z = xy$, then $[x, y] = z^2$ and $z^x = z^y = z^{-1}$. For any $k \geq 1$, we get by induction $[x,\, _ky] = z^{-(-2)^k}$ and $[y,\, _kx] = z^{(-2)^k}$. Therefore $[x,\, _{n-1}y], [y,\, _{n-1}x] \neq 1$ whereas $[x,\, _ny] = [y,\, _nx] = 1$. Thus $x$ and $y$ are mutually $n$-Engel elements. Furthermore, we have $G = \langle y, z\rangle$ and $[y,\, _2z] = [z,\, _ny] = 1$, so even $y$ and $z$ are mutually $n$-Engel elements. □

The following is an example obtained by GAP of a non-nilpotent group $G$ generated by two mutually 3-Engel elements, for which $\gamma_4(G)$ is abelian.

**Example 4.2.** Let $W = S_3 \, wr \, \mathbb{Z}_4$ be the wreath product of the symmetric group of degree 3 with the cyclic group of order 4. Thus, $|W| = 2^6 3^4$. We have $W = Q \ltimes N$, where $N$ is an elementary abelian group of order $3^4$ and $Q \simeq \mathbb{Z}_2 \, wr \, \mathbb{Z}_4$. Moreover, $Q$ is nilpotent of class 4. With the notation of GAP, let ele:=Elements($W$), $x :=$ ele[4] and $y :=$ ele[228]. Then $o(x) = o(y) = 4$ and $[x,\, _3y] = [y,\, _3x] = 1$. As $o(xy^{-1}) = 6$, the subgroup $G = \langle x, y\rangle$ of $W$ is not nilpotent. Finally, one can check that $G = S \ltimes N$ where $S$ is a group of order $2^5$ which is nilpotent of class 3.

For completeness reasons, we point out that $W = \langle x, y'\rangle$ with $y' :=$ ele[509] of order 6 and $[x,\, _3y'] = [y',\, _4x] = 1$. Hence, $W$ is a generated by two mutually 4-Engel elements and is not nilpotent. □

Notice that some more non-nilpotent groups generated by two mutually $n$-Engel elements can be found in the literature. For instance, Corollary 0.2 of [2] says that, for $n \geq 26$, the group $G(n) = \langle x, y \mid [x,\, _ny] = [y,\, _nx] = 1\rangle$ is not nilpotent. We can improve upon this. In fact, we show below that $G(4)$ is not soluble, because it has a quotient isomorphic to the symmetric group $S_8$.

**Example 4.3.** Let $S_8$ be the symmetric group of degree 8, and let $x = (1, 2, 3, 4)(5, 6)(7, 8)$ and $y = (1, 3)(2, 5)(4, 7, 6, 8)$. Put $x_n = [x,\, _ny]$ and $y_n = [y,\, _nx]$, for any $n \geq 0$ (so $x_0 = x, y_0 = y$). We then have:

$$\begin{aligned}
x_1 &= (1,6)(2,7)(3,8)(4,5) & y_1 &= (1,6)(2,7)(3,8)(4,5)\\
x_2 &= (1,5)(4,6) & y_2 &= (2,4)(5,7)\\
x_3 &= (1,5)(2,3)(4,6)(7,8) & y_3 &= (1,3)(2,4)(5,7)(6,8)\\
x_4 &= (1) & y_4 &= (1)\,.
\end{aligned}$$

In particular, $[x,\, _4y] = [y,\, _4x] = 1$. However $x$ and $y$ are of order 4, but $xy = (1, 5, 8, 6, 2)(3, 7, 4)$ is of order 15. The subgroup $G = \langle x, y\rangle$ is thus non-nilpotent. Using GAP, it is easy to see that $|G| = 8!$, so $G = S_8$. □

We now discuss the situation of Example 4.3. Clearly, if the pair $(x, y) \in G \times G$ satisfies the condition

$$[x,\, {}_4y] = [y,\, {}_4x] = 1, \tag{$*$}$$

then all conjugates $(x^g, y^g)$, for all $g \in G$, satisfy the analogous property. Therefore it is sensible to consider classes under conjugation.

It turns out by GAP that the only pairs $(x, y) \in G \times G$ satisfying $(*)$, that generate a non-nilpotent subgroup of $G$, have both $x$ and $y$ with cycle structure of type $(4)(2)(2)$ and, in addition, $x, y$ necessarily generate the whole group $G$. Without loss of generality, we may assume $x = (1, 2, 3, 4)(5, 6)(7, 8)$. For this $x$, we calculated all solutions $y \in G$ of $(*)$. We ended up with precisely 64 solutions. Of course, the group $C_G(x)$ of order 32 acts on the pairs of solutions. The stabilizer of this action is $C_G(x) \cap C_G(y) = Z(G) = 1$, so that we obtain two essentially distinct solutions.

**Other examples?** Suppose that in some finite group we can find Sylow $p$-subgroups $P, Q$ and elements $x \in P, y \in Q$ such that $[x, y] \in P \cap Q$. Let $c$ be the nilpotency class of $P$. Thus, $[x,\, {}_{c+1}y] = [y,\, {}_{c+1}x] = 1$. If $xy$ is not a $p$-element, then $\langle x, y \rangle$ is non-nilpotent. The groups in Examples 4.2 and 4.3 are of this form for $p = 2$. It would be very interesting to find analogous examples for all odd primes $p$.

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